The knot centre problem

Many decorative knots have been developed with a degree of spherical symmetry but most of them are formed around a solid core such as a glass sphere, or they have a central void. That leads to the question: Is it possible to find a symmetrical way to compose the centre of a ‘solid’ knot?

Basic centre

If we require the centre to be self holding rather than just a space filling bundle we can consider the centre as three perpendicular bundles of strands that pass through each other in such a way that each is held in place by the others, as shown in Figure 1.

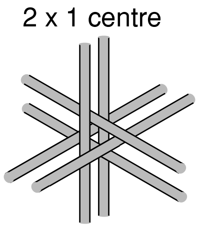

Figure 1: Each axis constrained by the others

In the general case there is an n x n+1 array of strands on each axis. Figure 2 shows the arrays for n=1 and n=2. The ellipses show end views of the bundles running parallel to the plane of the diagram

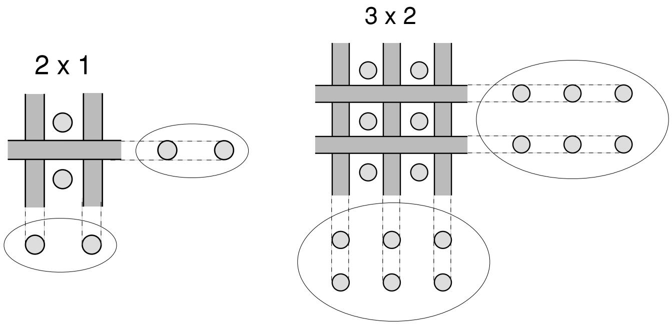

Figure 2: 2x1 and 3x2 centres

Connecting loops

The centre must be attached to the rest of the knot, and if we impose the requirement for the knot to be formed by one continuous strand then all the ‘ends’ coming out of the core must be connected to each other. The simplest way to do this is via simple loops. Further elaboration is possible to build up the outer layers of the knot but here we are concerned with the centre, where the outer complexity can be represented by a set of loops where each strand leaves the centre in one place and returns to it in another.

Consider two types of loop:

- Inter–axis loops that link together strands along different axes
- Same–axis loops that link one end of a strand with the other end of a (different) strand along the same axis

Looking down each axis we will in general see four sets of inter-axis loops as in Figure 3 (L) and looking between axes we will see three sets as in Figure 3 (R).

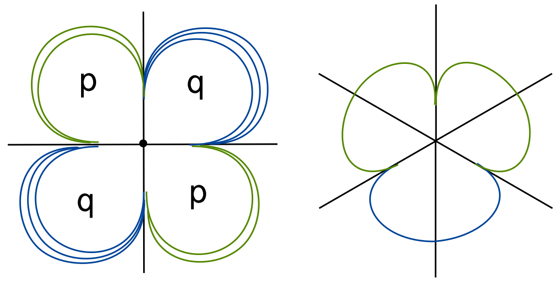

Figure 3: Inter axis loops, (L) Axial view, (R) Corner view

For each axis there are four possible positions for same-axis loops, in the spaces between the other two axes (one in each quadrant), see Figure 4 (L).. Looking between axes, as in Figure 4 (L), shows three sets, ie twelve possible positions in all.

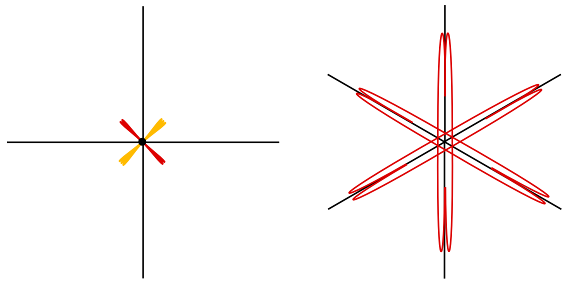

Figure 4: Axis self-loops, (L) One axis, (R) All axes

A basic 2 x 1 linking

Figure 5 shows a way to link all the strands of a 2x1 core with the minimum number of inter–axis links and no same–axis links. The diagram shows both ends of each pair of strands, and to help visualisation they are shown emerging through the six faces of an enclosing cube, with everything folded out flat. The ends marked x & x’ are the opposite ends of the same 2x1 bundle, likewise y & y’ and z & z’.

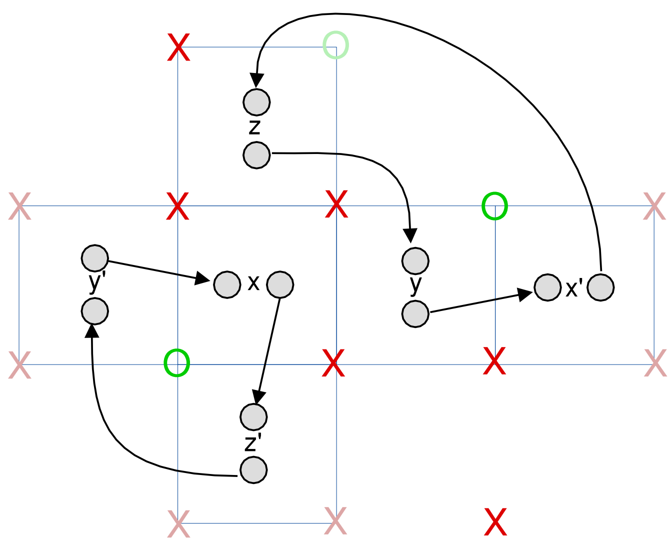

Figure 6: Asymmetry

Linking bundles of more than 2 x 1

A bundle of n+1 x n strands can always be divided into two equal sub bundles, see examples in Figure 7.

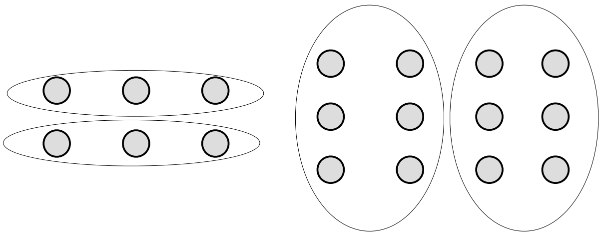

Figure 7: Equal sub bundles for 2x3 and 3x4

The 2 x 1 bundles can then be linked using the same way as the 2 x 1 case above, using one of two approaches:

- Same–axis loops reducing the number of ends that need to be connected
- Cycling strands within each bundle round different positions within the loops

Figure 8 shows how same axis loops can reduce the number of free ends. The number of loops needed is one less than the number of strands in the sub bundle – in this example 2 loops between 3 strands.

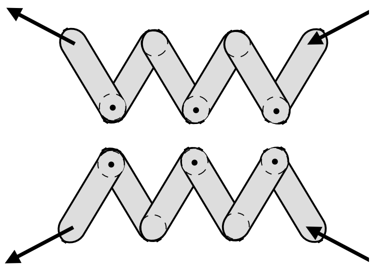

Figure 8: Bundle reduction with same–axis loops (2x3)

With the second approach, where whole sub bundles are linked through ‘trunking’ loops between axes, then to ensure that the whole is formed of a single strand making several passes round the circuit, an individual strand must occupy each position in the sub bundle in every loop. That will be true if:

- The positions of the strands within the sub bundle can be numbered in a systematic, cyclic way.
- In each loop each strand moves to the next numbered position.
- The number of loops and the number of strands in the sub bundle are co-prime.

There are 6 loops so the number of strands, ie n(n+1), must not be a multiple of either 2 or 3. By inspection, this requirement is first met is for a 10 x 11 centre, which has two 55 strand sub bundles. Others are: 13 x 14, 22 x 23, 25 x 26,34 x 35, 37 x 38,46 x 47, 49 x 50, etc.

Symmetry?

Symmetry is desirable, but what sort of symmetry, and can it be achieved?

The only centre configuration established is based on three orthogonal bundles of strands, and the loops that connect them, so let us consider the number and position of loops. Figure 9 shows the positions of all 12 possible inter-axis loops. The scheme above, used to link a 2 x 1 centre uses the six links shown in black but not the six shown in orange. It can be seen that the links form two circuits around opposite corners of the cube (shown green) and there are no comparable circuits round any of the other six corners.

Symmetry is therefor limited to a triad about the chosen diagonal (in this example the line -x = y = z)

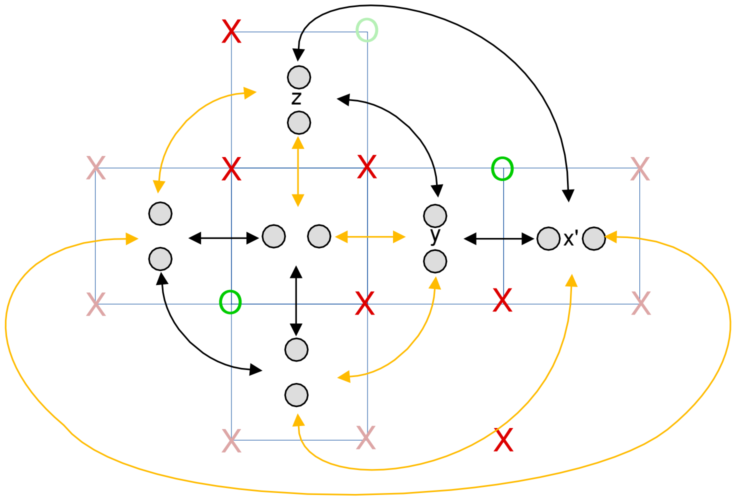

Figure 9: All possible inter-axis loops

Figure 10 shows a perspective view of the six loops, which form a two ended shape with three way rotational symmetry. For ease of visualisation the loops connecting each pair of axes (x–y, y–z, z–x) are coloured separately.

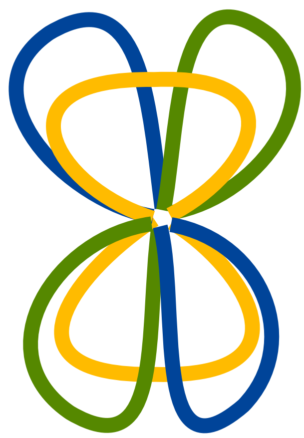

Figure 10: Perspective view of 2 x 1 loops

Is it possible to create a more symmetric pattern, for example one with loops in each of the 12 possible positions?

Suppose there are x loops in 12 positions. They will have 12x ends shared equally between the 6 bundle ends emerging from the centre, ie 4x ends each connected. To match this to the core bundle requires 4x = n(n+1). The only solutions are: x=3 (12 ends 4x3 array) & x=5 (20 ends, 5x4 array).

Different numbers of ends could be matched using same-axis loops. Can this be done with self loops in all possible positions, ie 4 per axis?

Suppose there are y same-axis loops in each position, (absorbing 4y ends per axis end), and x inter-axis loops (creating 4x ends per axis end). A match requires:

$$4x - 4y = n(n+1) \quad \text{ie} \quad x - y = n(n+1)/4.$$

The first few solutions (ends per axis end) are:

n=3 (12 ends), x=4 (16 ends), y=1 (–4 ends)

n=3 (12 ends), x=5 (20 ends), y=2 (–8 ends)

etc... increasing x & y in step.

Then (only giving the solution for y–1):

n=4 (20 ends), x=6 (24 ends), y=1 (–4 ends)

n=7 (56 ends), x=8 (32 ends), y=1 (–4 ends)

n=8 (32 ends), x=9 (36 ends), y=1 (–4 ends)

n=11 (44 ends), x=12 (48 ends), y=1 (–4 ends)

n=12 (48 ends), x=13 (52 ends), y=1 (–4 ends)

etc ...

Loop interactions

As well as a coherent central structure and a coherent way to join all the strands in the centre together with loops in the surrounding space, we also need to consider how the loops will interact with each other as they cross over in that space.

Figure 11 shows the twelve possible positions of inter-axis loops looking down one axis of the knot with a single loop in every possible position. The green and blue lops are edge on, two each in front of the plane of the diagram and two each behind.

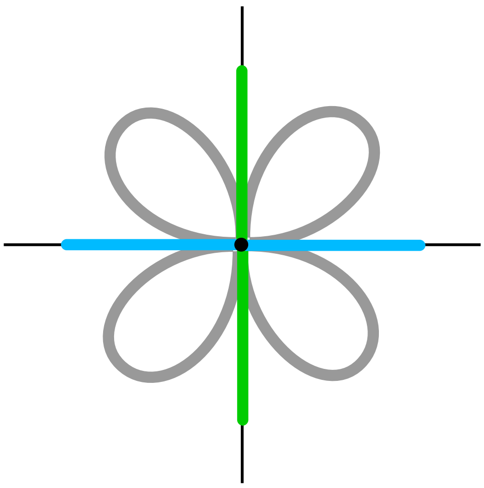

Figure 11: Axial view all inter-axis loop positions

These paths are essentially separate and don't intersect anywhere. The only touch where they meet to go through the core, and where the strands along each axis are grouped into two bundles, they will run alongside each other.

Figure 12 is a similar view down one axis showing all twelve possible positions of same-axis loops. The yellow loops are edge on. The red and orange loops are at 45° to the plane of the paper with the dark ones behind and the light ones in front. These loops do cross each other

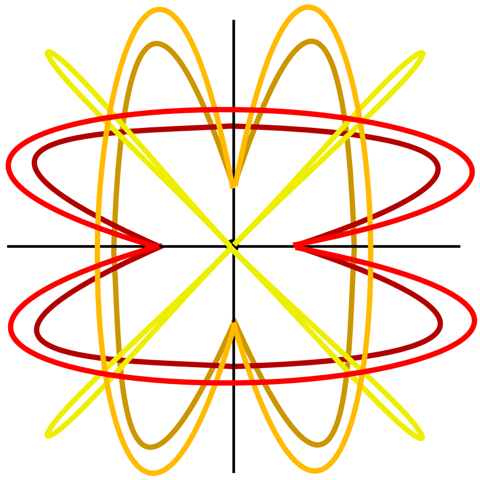

Figure 12:Axial view all same-axis loop positions

Figures 13 shows four same-axis loops emerging from the end of each axis. They are slightly offset for clarity at the overlaps.

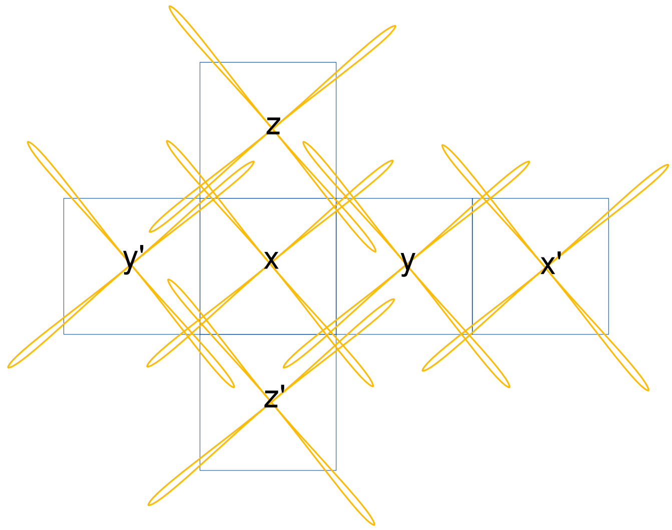

Figure 13: Interaction of same-axis loops

There are three loops at each overlap, and although they appear to do so at right angles in Figure 13 that is an artefact of the drawing being folded out flat. They actually intersect at 120° as shown in Figure 14, viewed between the axes, ie from a 'corner',

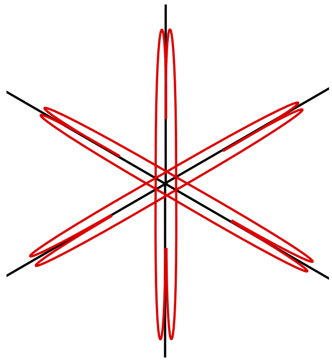

Figure 14: Corner view of same-axis loops

Figure 15 shows the two symmetrical ways that three loops can cross, with a left or right handed twist.

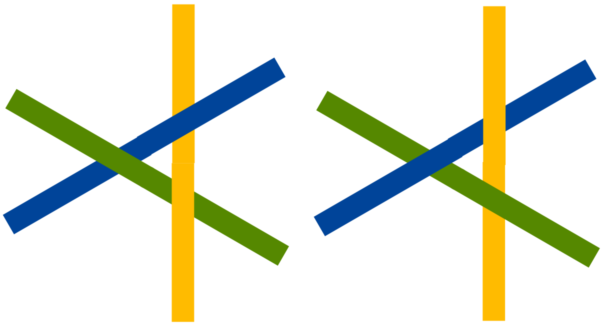

Figure 15: Left or right handed overlap

A symmetrical arrangement across the whole outer part of the knot can be achieved if adjacent corners have left and right handed overlaps, as shown in Figure 16.

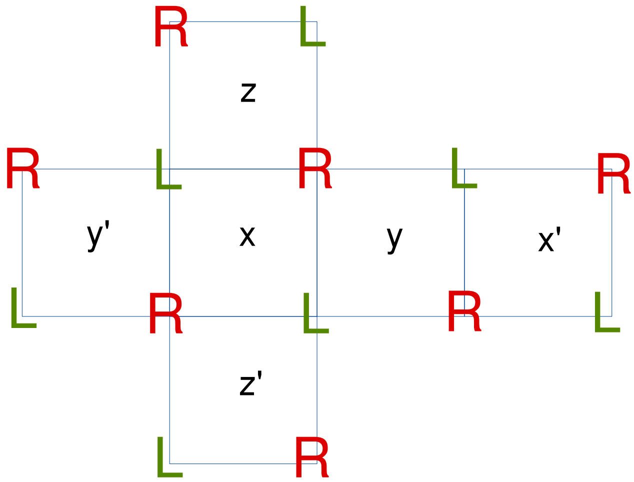

Figure 16: Distribution of loop overlap handedness

Figure 17 shows both inter-axis and same-axis loops, ie it merges Figures 11 & 12.

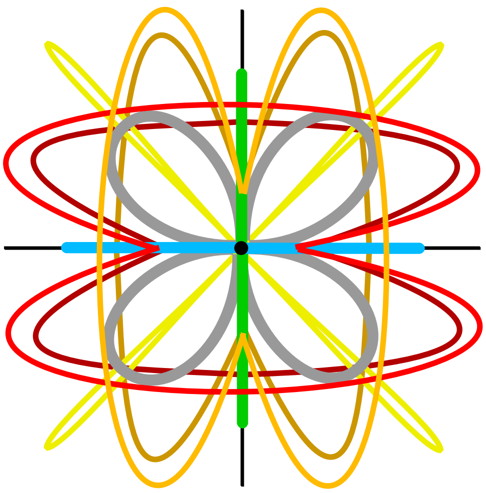

Figure 17: Axial view all loop positions

While this shows all the loops it is not easy to visualise how all the loops interact – in particular do the inter-axis and same-axis loops interact?

Inter axis loops link the ends of strands that emerge from the core 90° apart, whereas same-axis loops link ends that are 180° apart, ie on the opposite side. It would therefore be possible for the inter-axis loops to form an inner layer with the same-axis loops forming an outer layer.

Conclusions & unresolved questions

- Is it possible to find a symmetrical way to compose the centre of a 'solid' knot?
  Yes. Bundles of (n+1) x n strands on three orthogonal axis are mutually containing.
- Is it possible to link the ends of such bundles?
  Yes. A 2 x 1 array can be linked simply with inter-axis links and and larger array can be reduced to 2 x 1 by the use of same-axis loops.
- Are there any configurations where the bundles form inter-axis loops with strands cycling uniformly within them to achieve a single overall strand?
  Yes. For arrays of: 10 x 11, 13 x 14, 22 x 23, 25 x 26,34 x 35, 37 x 38,46 x 47, 49 x 50, etc
- Can the internal structure of cyclic bundles be arranged in an orderly way?
  Not investigated.
- Is the basic 2 x 1 linking structure symmetrical?
  Partly. It has three-way rotational symmetry, and inversion symmetry but it favours one particular direction, with links in some positions but not all.
- Are there any configurations that can be linked with the same number of inter-axis links in all positions?
  Yes. There are two cases: a 4 x 3 array and a 5 x 4 array.
- Are there any configurations that can be linked with the same number of inter-axis links in all positions and the same (maybe different) number of same-axis links in all positions?
  Yes. There are many cases, with centres: 4 x 3, 5 x 4, 8 x 7, 9 x 8, 12 x 11, 13 x 12, etc.
- Can intersecting loops do so symmetrically?
  Yes. Only same-axis loops intersect and they can do so with a right or left handed overlap.
- Can the handedness of overlap be symmetrical over the whole surface?
  Yes, with left and right handed overlaps alternating.
- Can inter-axis loops and same-axis loops fit tidily together?
  Yes, with one set inside the other.
- Can such knots be constructed in practice?
  The basic 2x1 centre could be constructed and linked but with no outer structure would not hold together. The more complex knots would be extremely challenging, maybe impractical.